MISCELLANEOUS.

68. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio State University, Athens, Ohio.

Find the locus of the vertex of the cone enveloping the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ so that the plane of contact will constantly touch $x^2+y^2+z^2=r^2$.

I. Solution by ELMER SCHUYLER, High Bridge, N. J.

From Aldis's *Solid Geometry*, equation of plane of contact to ellipsoid is

$$\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\gamma z}{c^2}-1=0,\ \frac{xx'}{r^2}+\frac{yy'}{r^2}+\frac{zz'}{r^2}-1=0,$$

equation of tangent to circle, $x'^2+y'^2+z'^2=r^2$.

$\therefore$ Since the plane of contact is tangent to sphere,

$$\frac{x'}{r^2}=\frac{\alpha}{a^2},\quad \frac{y'}{r^2}=\frac{\beta}{b^2},\quad \frac{z'}{r^2}=\frac{\gamma}{c^2} \text{ and } \left(\frac{\alpha r^2}{a^2}\right)^2+\left(\frac{\beta r^2}{b^2}\right)^2+\left(\frac{\gamma r^2}{c^2}\right)^2=r^2,$$

or equation is

$$\left(\frac{\alpha r}{a^2}\right)^2+\left(\frac{\beta r}{b^2}\right)^2+\left(\frac{\gamma r}{c^2}\right)^2=1,$$

which locus is an ellipsoid similar to the given one but with ratio of axes as $\sqrt{r}:1$.

II. Solution by W. B. CARVER, Senior Class, Dickinson College, Carlisle, Pa.

Let (x', y', z') be a point. Then $b^2c^2x'x+a^2c^2y'y+a^2b^2z'z=a^2b^2c^2$ is the equation of the plane of contact of the cone whose vertex is at (x', y', z') with the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

The condition that this plane touch the sphere $x^2+y^2+z^2=r^2$ is

$$b^4c^4x'^2+a^4c^4y'^2+a^4b^4z'^2=\frac{a^4b^4c^4}{r^2}.$$

Letting (x', y', z') move and x', y', and z' become variables, we have for the required locus

$$b^4c^4x^2+a^4c^4y^2+a^4b^4z^2=\frac{a^4b^4c^4}{r^2} \text{ or } \frac{x^2}{a^4/r^2}+\frac{y^2}{b^4/r^2}+\frac{z^2}{c^4/r^2}=1,$$

which is the equation of an ellipsoid.

III. Solution by the PROPOSER.

If (x', y', z') be the vertex of the cone, the plane of contact is

$$\frac{x'x}{a^2}+\frac{y'y}{b^2}+\frac{z'z}{c^2}-1=0\ldots\ldots(1);$$

and the sphere being $x^2+y^2+z^2-r^2=0\ldots\ldots(2)$, the condition that (2) is touched by (1) is

$$\frac{x'^2}{a^4}+\frac{y'^2}{b^4}+\frac{z'^2}{c^4}-\frac{1}{r^2}=0\ldots\ldots(3).$$

a concentric ellipsoid.

69. Proposed by WILLIAM SYMMONDS, A. M., Professor of Mathematics and Astronomy in Pacific College, Santa Rosa, P. O., Sebastopol, Cal.

Find the locus of a point equidistant from the circumferences of two fixed circles.

Solution by ELMER SCHUYLER, High Bridge, N. J.

Let radii be a and b, and $OO'=c$.

$OP=\sqrt{(x^2+y^2)}\ldots\ldots\ldots\ldots(1)$, $O'P=\sqrt{[y^2+(c-x)^2]}\ldots\ldots\ \ldots\ldots(2)$.

By condition, $O'P-OP=b-a$.

$\therefore \sqrt{[(c-x)^2+y^2]}-\sqrt{[x^2+y^2]}=b-a\ldots\ldots\ldots\ldots(3)$.

Clearing of fractions gives us $[c^2-(b-a)^2-2cx]^2=4(b-a)^2(x^2+y^2)\ldots(4)$.

This is a conic section and an ellipse, hyperbola, parabola (or particular case) according as $(b-a)^2[(b-a)^2-c^2]>$, $<$, or $=0$.

NOTE ON RIGHT TRIANGLES.

Every right-angled triangle has two concealed roots. By three different combinations of the two roots, the three sides are formed. The longest side is the sum of the squares of the two roots. The second side is the difference of the squares of the two roots. The third side is twice the product of the two roots.

The perimeter is equal to twice the greater root multiplied by the sum of the two roots. The area is equal to the product of the two roots multiplied by the product of the sum and difference of the two roots.

A prime right-angled triangle is one whose sides are integral and cannot all be divided by the same number without a remainder. A prime triangle is the result of having one of the roots odd and one even. Exception.—If the even root is just twice the odd root, the resulting triangle will not be prime, as its sides will all be divisible by the square of the odd root.

If both the roots be odd or both even the sides of the triangle will be divisible by two and the triangle will not be prime. Any odd number may be one side of a prime triangle; in many cases the same odd number will serve as one side of several different prime triangles, as for example, 13, 12, 5, and 13, 84, 85. Any even number divisible by 4 can be one side of a prime triangle as a prime triangle has always one even side. The digit, 2, must be a factor twice, or both the roots will be odd numbers. The same even number may be a side of several prime triangles, as 12, 13, 5, and 12, 35, 37.

A prime right-angled triangle has two of its sides expressed by odd numbers. Find the sum and the difference of these. Each will be the double of a perfect square. The square root of one-half the sum will be the greater root of